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A characterization of level planar graphs

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Abstract

We present a characterization of level planar graphs in terms of minimal forbidden subgraphs called minimal level non-planar (MLNP) subgraph patterns. We show that an MLNP subgraph pattern is completely characterized by either a tree, a level non-planar cycle or a level planar cycle with certain path augmentations.

Keywords: Level graphs; Minimal non-planarity; Maximum level planar subgraph

1. Introduction

Level graphs are an important class of graphs that generalize bipartite graphs. Such graphs are used to model hierarchical relationships or workflow diagrams. From the point of view of representing the relationships graphically, it is desirable that the graphs are drawn with as few edge crossings as possible. Jünger et al. [4] have presented a linear-time algorithm to determine if a level graph is planar; this algorithm can be modified to determine a planar embedding of the graph as well.

It is in the case of level graph non-planarity that we are interested and, in particular those that are *minimally* non-planar. A level graph is minimal non-planar if the removal of *any* edge makes the resulting level graph planar. Such graphs, then, are the analogue of K_5 and $K_{3,3}$ in the case of general graph planarity. The characterization we provide

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is of interest for two reasons. Firstly, it contributes to the understanding of level graphs and is an interesting contribution to that body of work in its own right; secondly, the characterization that we provide has the potential to be of use in an Integer Linear Programming formulation of the Maximum Level Planar Subgraph problem, where cutting planes based on minimal level non-planar (MLNP) subgraphs prove to be useful [5]. The Maximum Level Planar Subgraph problem is to determine the maximum level planar subgraph of a level non-planar graph.

This paper is organized as follows. After summarizing the necessary preliminaries in the next section, we introduce the MLNP subgraph patterns in Section 3. We prove that the patterns are MLNP and that the set of patterns is complete for the special case of hierarchies. This result is generalized to level graphs in Section 4. We place this work in context in Section 5.

2. Preliminaries

We begin this section with some definitions and then we describe Di Battista and Nardelli's characterization of level non-planar (LNP) hierarchies.

A *level graph* $G = (V, E, \lambda)$ is a directed acyclic graph with a mapping $\lambda : V \rightarrow \{1, 2, \ldots, k\}$, $k \geqslant 1$, that partitions the vertex set V as $V = V_1 \cup V_2 \cup \cdots \cup V_k$, $V_j = \lambda^{-1}(j)$, $V_i \cap V_j = \emptyset$ for $i \neq j$, such that $\lambda(v) = \lambda(u) + 1$ for each edge $(u, v) \in E$. A vertex $v \in V_j$ is called a *level-j vertex* and V_j is called the jth *level* of G.

A drawing of a level graph G in the plane is a *level drawing* if the vertices of every V_j, $1 \leqslant j \leqslant k$, are placed on a horizontal line $l_j = \{(x, k - j) \mid x \in \mathbb{R}\}$, and every edge $(u, v) \in E$, $u \in V_j$, $v \in V_{j+1}$, $1 \leqslant j < k$, is drawn as a straight line segment between the lines l_i and l_{i+1}. A level drawing of G is called *level planar* if no two edges cross except at common endpoints. A level graph is *level planar* if it has a level planar drawing.

A *hierarchy* is a level graph $G(V, E, \lambda)$ where for every $v \in V_j$, $j > 1$, there exists at least one edge (w, v) such that $w \in V_{j-1}$. That is, all sources appear on the first level.

The characterization of level non-planarity by patterns of subgraphs has been suggested earlier by Di Battista and Nardelli who have identified three (not necessarily minimal) patterns of LNP subgraphs for hierarchies [1]. We call these LNP patterns. To describe the LNP patterns, we give some terminology similar to theirs. A *path* is an ordered sequence of vertices $(v_1, v_2, \ldots, v_n)$, $n > 1$ such that for each pair (v_i, v_{i+1}), $i = 1, 2, \ldots, n-1$ either (v_i, v_{i+1}) or (v_{i+1}, v_i) belongs to E. Let i and j, $i < j$ be two levels of a level graph $G = (V, E, \lambda)$. LACE(i, j) denotes the set of paths $\mathscr{C}$ connecting any two vertices $x \in V_i$ and $y \in V_j$ such that for any $z \in C', C' \in \mathscr{C}$, $z \in V_t$, $i \leqslant t \leqslant j\}$. If C_1 and C_2 are completely distinct paths belonging to LACE(i, j) then a *bridge* is a path connecting vertices $x \in C_1$ and $y \in C_2$ traversing only vertices in $V_i \cup V_{i+1} \cup \cdots \cup V_j$. Vertices x and y are thus called the *endpoints* of a bridge. The next theorem gives a characterization of LNP patterns for hierarchies, as opposed to level graphs.

Theorem 1 (Di Battista and Nardelli [1]). *Let $G = (V, E, \lambda)$ be a hierarchy with $k > 1$ levels. G is level planar if and only if there is no triple $L_1, L_2, L_3 \in$ LACE(i, j), $0 < i <$*

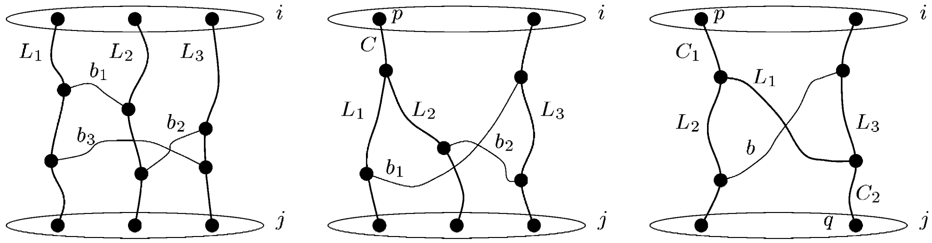

Fig. 1. Di Battista-Nardelli's LNP patterns.

$j \leqslant k$, that satisfies one of the following conditions:

(a) *L_1, L_2 and L_3 are completely disjoint and pairwise connected by bridges. Bridges do not share a vertex with L_1, L_2 and L_3, except in their endpoints* (*see Fig.* 1(*a*));

(b) *L_1 and L_2 share an endpoint p and a path C* (*possibly empty*) *starting from p, $L_1 \cap L_3 = L_2 \cap L_3 = \emptyset$; there is a bridge b_1 between L_1 and L_3 and a bridge b_2 between L_2 and L_3, $b_1 \cap L_2 = b_2 \cap L_1 = \emptyset$* (*see Fig.* 1(*b*));

(c) *L_1 and L_2 share an endpoint p and a path C_1* (*possibly empty*) *starting from p; L_1 and L_3 share an endpoint q ($q \neq p$) and a path C_2* (*possibly empty*) *starting from q, $C_2 \cap C_1 = \emptyset$; L_2 and L_3 are connected by a bridge b, $b \cap L_1 = \emptyset$* (*see Fig.* 1(*c*)).

3. MLNP patterns in hierarchies

Di Battista and Nardelli, as we have remarked, have identified three level non-planar patterns for hierarchies. From the recognition of an LNP pattern in a hierarchy, one can decide that the hierarchy is not level planar, but one cannot guarantee that a removal of an edge from the subgraph matching an LNP pattern leads to the level planarity of the graph.

MLNP patterns are defined to have the following property: If a level graph $G = (V, E, \lambda)$ matches an MLNP pattern, then any subgraph $G' = (V, E', \lambda)$ of G, with $E' = E \setminus \{e\}, e \in E$, is embeddable without crossings on levels. The MLNP patterns are divided into three categories: trees, LNP cycles, and level planar cycles with incident paths. We give a comprehensive description of each of these categories and show that the categories are complete for hierarchies.

The terminology that is used to describe the MLNP patterns is compatible with Harary [3], except that we denote by a *chain* a tree $T(V,E)$ where $E = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{|V|-1}, v_{|V|})\}$. Furthermore, we define some terms that are common to all of the patterns. The upper- and lower-most levels that contain vertices of a pattern P are called *extreme levels* of P. The extreme levels of a pattern are not necessarily the same as the extreme levels 1 and k of the input graph G. If pattern P is instantiated in G as subgraph G_P and the highest level of G_P in G is i and the lowest level is j ($i < j$) then the extreme levels of P correspond to levels i and j in G. If vertex v lies on an extreme level then we call this extreme level the *incident* extreme level and the other extreme level the *opposite* extreme level of v.

3.1. Trees

Characterization: We can characterize an MLNP tree pattern as follows. Let i and j be the extreme levels of a pattern and let x denote a root vertex with degree 3 that is located on one of the levels $i, \ldots, j$. From the root vertex emerge 3 subtrees that have the following common properties (see Fig. 2 for illustrations of two typical patterns):

- each subtree has at least one vertex on both extreme levels;
- a subtree is either a chain or it has two branches which are chains;
- all the leaf vertices of the subtrees are located on the extreme levels, and if there is a leaf vertex v of a subtree S on an extreme level $l \in \{i, j\}$ then v is the only vertex of S on the extreme level l;
- those subtrees which are chains have one or more non-leaf vertices on the extreme level opposite to the level of their leaf vertices.

The location of the root vertex distinguishes the two characterizations.

(T1) The root vertex x is on an extreme level $l \in \{i, j\}$ (see Fig. 2(a)):
- at least one of the subtrees is a chain starting from x, going to the opposite extreme level of x and finishing on x's level;

(T2) The root vertex x is on one of the intermediate levels l, $i < l < j$ (see Fig. 2(b)):
- at least one of the subtrees is a chain that starts from the root vertex, goes to the extreme level i and finishes on the extreme level j;
- at least one of the subtrees is a chain that starts from the root vertex, goes to the extreme level j and finishes on the extreme level i.

Theorem 2. *A subgraph matching either of the two tree characterizations T1 or T2 is MLNP.*

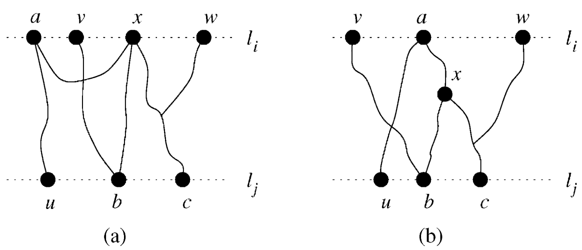

Fig. 2. MLNP trees.

Proof. The proof of level non-planarity of T1 and T2 is straightforward by matching T1 and T2 to the LNP pattern (a).

To prove minimality, we consider the two forms of the tree patterns separately. Consider $T1$ where the root vertex x and vertices a, v and w of the chains are located on the same extreme level. Every subtree detached from the root-vertex x has a level planar layout. Thus if we remove one of the edges incident upon the leaf vertices on the extreme level of the root vertex (like the vertices v or w in Fig. 2(a)) then the corresponding subtree can be embedded under the root vertex x and between the other subtrees without any crossings. If we remove an edge incident upon the leaf vertices near the opposite level of the root vertex (for example, the path from vertex c to the branching point in Fig. 2(a)) then the modified subtree can be embedded on top of the chain-shaped subtree (according to the characterization there has to be one). Next, if we remove any other edge, we will have two disconnected subgraphs: one which contains the root vertex and the other which does not contain the root vertex. The former is a reduced case of the removal of an edge incident to a leaf vertex and the other component can be embedded.

In the case T2 when the root vertex is not on an extreme level, we consider two cases: the removal of an edge connecting the leaf vertex of a chain and the removal of an edge connecting a leaf vertex of a non-chain subtree. In the former case, the two chain subtrees can be embedded on top of each other. In the latter case, the path can be embedded under or on top of a chain by repositioning either vertices v or u as appropriate. If we remove any other edge then, again, we will have two disconnected subgraphs from which the subgraph containing the root vertex is a reduced case of the removal of an edge incident to a leaf vertex and the other subgraph can be embedded. □

The following three lemmas and the next theorem prove that the two tree patterns in our characterization are unique.

Lemma 3. *If LNP pattern* (*a*) *matches a tree then each one of the paths* L_1, L_2, L_3 *contains only one vertex being the end vertex of a bridge.*

Proof. Each path L_i of LNP pattern (a) has at most two vertices where the bridges are connected. Suppose the lemma is not true. Then at least one of the paths, say L_1, contains two distinct vertices c_1 and c_2 connecting bridges. By hypothesis there must exist a path S between c_1 and c_2 along the bridges and at least one of the other two paths, L_2, L_3. But there is also a path T between c_1 and c_2 along L_1. The paths S and T constitute a cycle which contradicts our tree requirement. □

Lemma 4. *If LNP pattern* (*a*) *matches a tree then its bridges must form a subgraph homeomorphic to* $K_{1,3}$.

Proof. From the previous lemma, each one of the paths L_1, L_2 and L_3 must have exactly one vertex connecting to a bridge. Then, the pattern has, in total, three vertices c_1, c_2 and c_3 to connect bridges, b_i. These vertices must be distinct since the pattern

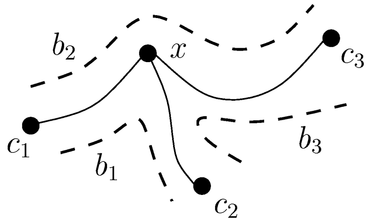

Fig. 3. Relationship of bridge connection vertices, c_i, and bridges, b_i.

requires that the paths L_i do not have any common vertices. Also, the pattern requires that the bridges have common vertices with the paths only in their ends. Hence, we need to construct a tree that connects the vertices c_1, c_2 and c_3 so that each of the paths $(c_1,\dots,c_2)$, $(c_1,\dots,c_3)$ and $(c_2,\dots,c_3)$ does not go through the third vertex of the set $\{c_1,c_2,c_3\}$. The only possibility for such a graph is the presence of a fourth vertex $x \notin \{L_1,L_2,L_3\}$ connected to each of the vertices c_1, c_2 and c_3, forming a graph homeomorphic to $K_{1,3}$. □

Fig. 3 illustrates the relationship between bridge connection vertices c_i and the bridges, b_i.

Lemma 5. *Only LNP pattern* (*a*) *can be matched to a tree.*

Proof. Suppose the lemma is not true. Consider LNP pattern (b). To avoid a cycle in pattern (b), there must be only one vertex x connecting bridges on the path L_3. (This can be proven analogously to Lemma 3.) Also, the branching vertex v of the paths L_1 and L_2 and the vertices y and z connecting bridges on these paths must be non-distinct, otherwise there will be a cycle with the sequence of vertices (x,y,v,z,x). But then, the two bridges connect the same pair of vertices (x and $y=v=z$) and do not cause level non-planarity.

For pattern (c), the proof is analogous. Let v be the branching vertex of L_1 and L_2, and u, the branching vertex of L_1 and L_3. Let x be the vertex of L_2 where the bridge connects to L_2 and likewise, y and L_3. To avoid cycles, we have to collapse vertices u and x and vertices v and y. But then, the path L_1 and the bridge connect the same pair of vertices and do not cause level non-planarity. □

Fig. 4(b) illustrates this situation.

Theorem 6. *Let T be a tree. T is MLNP if and only if it matches either of the two tree characterizations.*

Proof. From the previous lemmas, it is possible to derive an LNP tree pattern (not necessarily minimal) from LNP pattern (a) only. Consider a tree matching LNP pattern (a). If the pattern is bounded by levels i and j, but the vertices of bridges occur on levels $l_1,\dots,l_2$, where $i<l_1$ and $l_2<j$ then we can remove all the edges of the

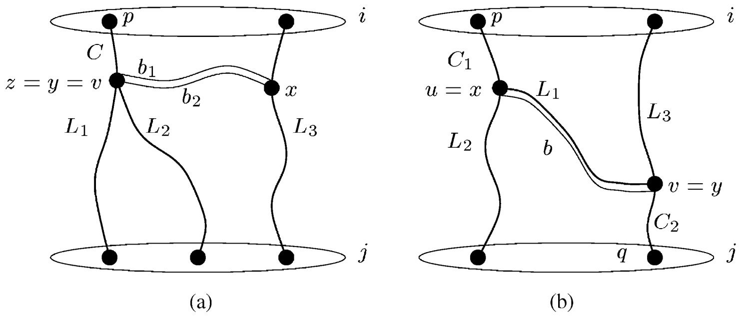

Fig. 4. Lemma 5: LNP patterns (b) and (c) cannot match a tree.

paths L_i which connect vertices on levels $i,\dots,l_1$ and $l_2,\dots,j$ without affecting level planarity. Moreover, we can narrow the range of levels $l_1,\dots,l_2$ even more, until both levels l_1 and l_2 contain at least one vertex v whose degree in the subgraph bounded by levels l_1 and l_2 is greater than 1. Then, it can be shown that the tree between levels $l_1,\dots,l_2$ is homeomorphic to either of the MLNP tree patterns. From Lemmas 3 and 4 each of the paths L_i has exactly one vertex c_i to connect a bridge and the bridges form a subgraph homeomorphic to $K_{1,3}$. Consequently, after narrowing the levels to $l_1,\dots,l_2$, each of the new extreme levels l_1, l_2 contains at least one of the following:

- a root vertex (x);
- a vertex of a path from x to c_i (c_i included).

In the latter case, if the vertex, say d, on level l_1 or l_2 is not identical to vertices x or c_i, we can remove the part of the upward path L_i from the extreme level of d to the vertex c_i. The tree maintains level non-planarity since the path L_i of LNP pattern (a) starts from the vertex d now. After performing this operation on each path L_i, we obtain a tree that matches either of our characterizations. □

3.2. Cycles

We now consider cycles that are bounded by the extreme levels of the pattern. We will consider two types of cycle: firstly, an LNP cycle, and then, cycles that are level planar but that are augmented by one or more paths that result in level non-planarity.

A cycle must then contain at least two distinct paths between the extreme levels having vertices of the extreme levels only in their endpoints. These paths are called *pillars*.

3.2.1. LNP cycles

Theorem 7. *If a cycle has more than two distinct paths connecting the vertices on the extreme levels of a pattern, it is MLNP.*

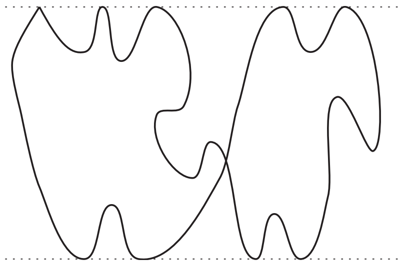

Fig. 5. An LNP cycle.

Proof. The number of such paths must be even. So, following our assumption of more than two paths, the number of paths must be at least 4 in an LNP cycle. Without loss of generality, consider the 4-path case first. Let the extreme levels be i and j. Let us denote a sequence of paths along the cycle $A=(v_a,\dots,v_b)$, $B=(v_b,\dots,v_c)$, $C=(v_c,\dots,v_d)$, $D=(v_d,\dots,v_a)$, and $v_a,v_c \in V_i$, $v_b,v_d \in V_j$. Consider LNP pattern (c). The paths A, B, C can be mapped always to the paths L_2, L_1, L_3 of the pattern, respectively. Then the remaining path D can be mapped to the bridge in LNP pattern (c). If the number of paths is greater than 4, the first three paths can be mapped as in the case of 4 paths, and the remaining paths can be mapped to the bridge. □

Such a cycle is minimal since any edge that is removed from an LNP cycle results in a chain that can be drawn as a level planar graph (Fig. 5).

3.2.2. Level planar cycles

Level planar cycles can be augmented by a set of chains to obtain minimal level non-planarity. First, we give some terminology related to level planar cycles. A vertex that lies on a pillar is called an *outer* vertex; all the remaining vertices are *inner* vertices. The endpoints of pillars are *corner* vertices; if an extreme level i has only one vertex it is called a *single corner* vertex. A *bridge* in the context of a planar cycle is the shortest walk between corner vertices on the same level; a bridge contains two corner vertices as its endpoints and the remainder are inner vertices. A pillar is *monotonic* if, in a walk of the cycle, the level numbers of subsequent vertices of the pillar are monotonically increasing or decreasing, depending on the direction of traversal. We call two paths or chains *parallel* if they start on the same pillar and end on the same extreme level. If a chain is connected to a cycle by one of its vertices having degree 1 (considering only edges of the chain), then this vertex is called the *starting vertex* of the chain and the level where this vertex lies, the *starting level*. The other vertex of degree 1 of the chain is then the *ending vertex* and corresponding level, the *ending level*.

Characterization: Given a level planar cycle whose extreme levels are i and j, there are four cases to consider where augmentation of the level planar cycle by paths results in minimal level non-planarity. The pattern cannot contain one of the tree patterns given

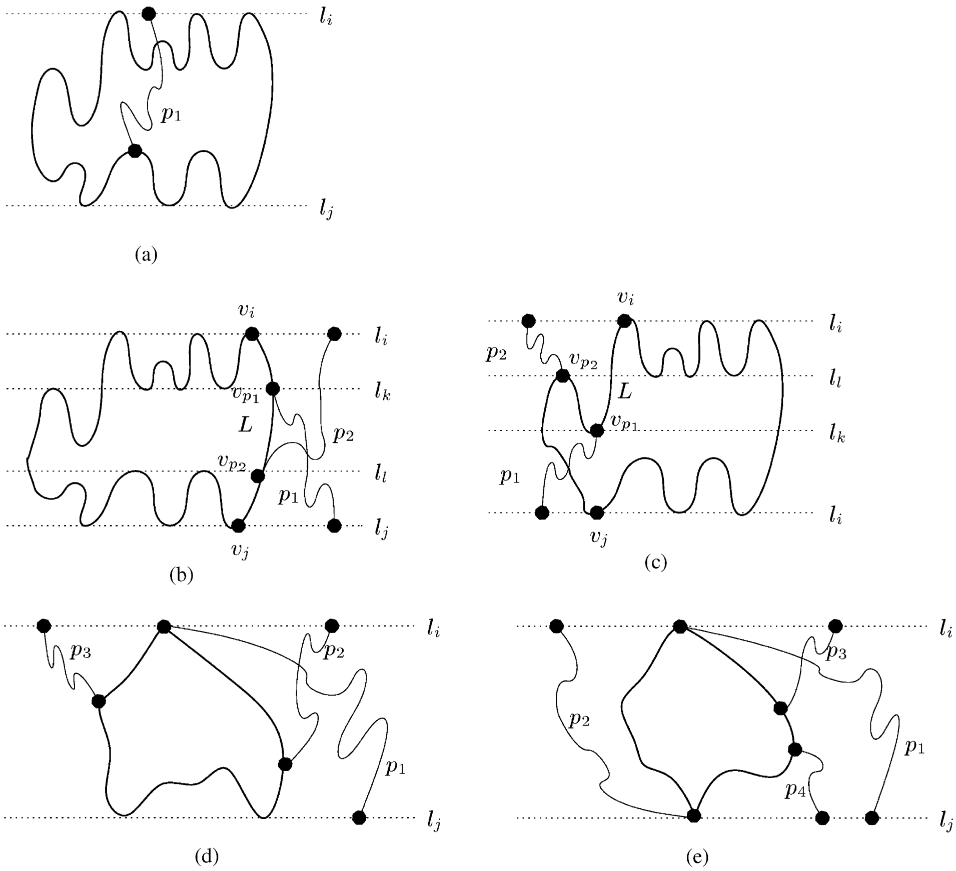

Fig. 6. Level planar cycles with paths.

earlier. We enumerate these augmenting paths below. In all cases the paths start at a vertex on the cycle and end on an extreme level.

(C1) A single path p_1 starting from an inner vertex and ending on the opposite extreme level of the inner vertex; p_1 and the cycle share only one vertex. The path will have at least one vertex on an extreme level, the end vertex, and at most two, the start and end vertices. An example of this is illustrated in Fig. 6(a).

(C2) Two paths p_1 and p_2, starting, respectively, from vertices v_{p_1} and v_{p_2}, $v_{p_1} \neq v_{p_2}$, of the same pillar $L = (v_i, \dots, v_{p_1}, \dots, v_{p_2}, \dots, v_j)$ terminating on extreme levels j and i, respectively. Vertices v_{p_1} or v_{p_2} may be identical to corner vertices of L ($v_{p_1} = v_i$ or $v_{p_2} = v_j$) only if the corner vertices are not single corner vertices on their extreme levels. Paths p_1 and p_2 do not have any vertices on the extreme levels other than their start (if corner) and end vertices. There are two subcases according to the levels of v_{p_1} and v_{p_2}:

- $\lambda(v_{p_1}) < \lambda(v_{p_2})$;
- $\lambda(v_{p_1}) \geqslant \lambda(v_{p_2})$, in which case L must be a non-monotonic pillar.

Figs. 6(b) and (c) illustrate typical subgraphs matching the two subcases, respectively.

(C3) Three paths, p_1, p_2 and p_3. Path p_1 starts from a single corner vertex and ends on the opposite extreme level; paths p_2 and p_3 start from opposite pillars and end on the extreme level where the single corner vertex is. Neither p_2 nor p_3 can start from a single corner vertex. Fig. 6(d) illustrates a level planar cycle augmented by three paths causing level non-planarity.

(C4) Four paths, p_1, p_2, p_3 and p_4. The cycle comprises a single corner vertex on each of the extreme levels. Paths p_1 and p_2 start from different corner vertices and end on the opposite extreme level to their start with the paths embedded on either side of the cycle such that they do not intersect; paths p_3 and p_4 start from distinct non-corner vertices of the same pillar and finish on different extreme levels. The level numbers of starting vertices are such that they do not cause crossing of the last two paths. See Fig. 6(e) for an illustration.

We will now prove that each of the path-augmented cycles is MLNP and, in Theorems 9–13, prove that this set is complete for hierarchies.

Theorem 8. *Each of the four path-augmented cycles is MLNP.*

Proof. The augmented cycles are LNP because each can be mapped straightforwardly to one of Di Battista and Nardelli's LNP patterns. To see minimality we consider the three cases of the starting position of the path-augmentation on the cycle.

Suppose the start vertex is an inner vertex of a cycle. Since no subgraph matches an MLNP tree, breaking either an edge of the path or an edge of the cycle yields a level planar embedding.

In case of a path-augmented cycle of type C2, the removal of any edge of the cycle allows one of the augmenting paths to be embedded through the "gap" left by that edge. The removal of any edge of an augmenting path allows that path to be embedded on the internal face of the cycle. In both cases no crossings will remain. □

For paths starting from corner vertices similar reasoning holds.

Theorem 9. *If an MLNP graph G comprises a level planar cycle and a single path p_1 connected to the cycle, then p_1 starts from an inner vertex of the cycle and ends on the opposite extreme level.*

Proof. For a path to cause level non-planarity, the path must start from an inner vertex of the cycle. Otherwise, the path can be embedded on the external face. There are only two possibilities for causing level non-planarity: crossing with the incident bridge or, crossing with the opposite bridge or one of the pillars. In the former, the path in combination with the lower part of the cycle forms an LNP tree. Since this LNP tree is minimal, the combination of the cycle and the path is not minimal. Therefore, the latter is the only remaining MLNP case. □

Theorem 10. *If an MLNP graph G comprises a level planar cycle and two paths p_1 and p_2 connected to the cycle, then p_1 and p_2 start from the same pillar and end on an extreme level and either they cross or they start from a non-monotonic sub-chain of the pillar.*

Proof. Neither of the two paths may start from an inner vertex of the cycle because otherwise either they can be embedded on the internal face, or at least one of them matches type C1 above, or they form an LNP tree. Since the latter two cases are not minimal both paths must start from a pillar. Moreover, they must start from the same pillar, otherwise, both paths can be embedded on the external face. The paths must finish on extreme levels, otherwise they can be embedded on the internal face. Moreover, the extreme levels must be different for if they are the same, the pattern—although it can be made LNP by introducing non-monotonic paths and a non-monotonic pillar—will not be minimal since it can be shown to match an MLNP tree pattern. If the extreme levels are different, then either the paths cross or there is a non-monotonic pillar that causes a crossing of the cycle and a path. □

Theorem 11. *If an MLNP graph G comprises a level planar cycle and three paths p_1, p_2 and p_3 connected to the cycle, then G has a single corner vertex c_1 with p_1 starting at c_1 and extending to the opposite extreme level and p_2 and p_3 starting on opposite pillars and ending on the extreme level that contains c_1.*

Proof. As in the case of two paths, none of the paths may start from an inner vertex. Hence, all the paths should start from pillars. Additionally, all the paths must end on extreme levels, otherwise, they can be embedded on the internal face. No pair of paths can create minimal level non-planarity of the type C2 above. These conditions are met if one of the paths starts from a single corner vertex. If there were no other paths, the path starting from the single corner could be embedded on the external face on both sides of the cycle. However, if we have two paths starting from different pillars, not from a single corner vertex, and ending on the extreme level of the single corner vertex, a level planar embedding is not possible. □

Theorem 12. *If an MLNP graph G comprises a level planar cycle and four paths $p_1, \ldots, p_4$ connected to the cycle, then G has two single corner vertices c_1 and c_2 with p_1 starting at c_1 and extending to the opposite extreme level of c_1, p_2 starting at c_2 and extending to the opposite extreme level of c_2, and p_3 and p_4 starting on the same pillar and diverging to end on opposite extreme levels such that they do not cross.*

Proof. This is proved analogously to the previous theorem, considering two corner vertices instead of one. □

Theorem 13. *If an LNP graph G comprises a level planar cycle and five or more path augmentations that extend to extreme levels, then G cannot be MLNP.*

Proof. An LNP pattern with two parallel paths cannot be minimal (since any path that crosses one will cross both or one of the parallel paths is redundant) unless one of the parallel paths can be embedded on the other side of the cycle, in which case this path starts from a single corner vertex.

Suppose we have an MLNP graph comprising a level planar cycle and five path augmentations extending to extreme levels. Since removing any edge from an MLNP graph makes it planar, there must be four non-crossing paths. This can be achieved only by having on each pillar either two diverging paths ending on opposite extreme levels or, parallel paths where one pair of paths starts out from a single corner vertex. In neither case is it possible to add a fifth path so that minimal level non-planarity holds. □

3.3. MLNP subgraphs in hierarchies

Having shown that our characterizations of trees, LNP cycles and path-augmented cycles are minimally level non-planar, it only remains for us now to show that this set is a complete characterization of MLNP subgraphs.

Theorem 14. *The set of MLNP patterns characterized by trees, LNP cycles and path-augmented level planar cycles of Sections* 3.1, 3.2.1, *and* 3.2.2, *respectively, is complete for hierarchies.*

Proof. Every graph comprises either a tree, or one, or more, cycles. It remains to prove that there is no MLNP pattern containing more than one cycle. Suppose a graph is MLNP and it has more than one cycle. Then it must be a subcase of one of Di Battista and Nardelli's LNP patterns. Each of these, however, has at most one single cycle and the remainder of the patterns comprises chains. Then at least one of our cycles must be broken in order to match it to a chain, thus contradicting the hypothesis. □

4. MLNP subgraphs in level graphs

We have given in the previous section a characterization of level planar hierarchies in terms of minimal forbidden subgraphs. It remains to show that the described patterns characterize level planarity for general level graphs as well.

Theorem 15. *Let $G=(V,E,\lambda)$ be a level graph with $k>1$ levels. Then G is not level planar if and only if it contains one of the MLNP patterns as described in Sections* 3.1, 3.2.1, *and* 3.2.2.

Proof. If a subgraph G_p of G corresponds to an MLNP pattern, then G must be LNP.

It remains to prove the opposite direction. Suppose there exists a minimal pattern P of level non-planarity that is not applicable for hierarchies. Let G be a level graph such that P is the only pattern of level non-planarity in G.

Since G is not level planar, augmenting the graph by an incoming edge for every source preserving the leveling in the graph constructs an LNP hierarchy $H = (V, E \cup E_H, \lambda)$, where E_H is the set of all extra added edges. Let $\mathscr{P}$ be the set of all subgraphs of H corresponding to MLNP patterns. By assumption there exists an edge $e_p \in E_p \cap E_H$ for any $G_p \in \mathscr{P}$, $G_p = (V_p, E_p, \lambda)$. Removing the edge e_p from H for every $G_p \in \mathscr{P}$, we construct a level planar graph H'. By construction, H' contains G as a subgraph. Since every subgraph of a level planar graph must be level planar itself, this contradicts G being an LNP subgraph. □

5. Conclusion

We have given a characterization of level planar graphs in terms of minimal forbidden subgraphs. Characterization of families of graphs by forbidden minors or, obstructions, is a common technique for undirected graphs that are closed under minor ordering. We have shown that for the family of level planar (directed) graphs, the set of obstructions is small.

This description of level planarity is an important contribution to solving the $\mathscr{N}\mathscr{P}$-hard level planarization problem [2] for practical instances. Based on the characterization of level planar graphs an integer linear programming formulation for the level planarization problem can be given, supporting the study of the associated polytope for the development of an efficient branch-and-cut algorithm.

Developing efficient algorithms to detect MLNP subgraphs or classes of them is an interesting and, as yet, unsolved problem.

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